MATH 347: FUNDAMENTAL MATHEMATICS, FALL 2015

PRACTICE PROBLEMS FOR CHAPTER 14 AND THEIR SOLUTIONS

1. Let $(x_n)_n$ be a sequence and a, b be reals with $a < b$. Suppose that for each $N \in \mathbb{N}$ there is $n \geq N$ such that $x_n \in [a, b]$. Prove that $(x_n)_n$ has a convergent subsequence whose limit is in $[a, b]$.

Solution. Note that choosing increasing values of N we can get many indices n for which $x_n \in [a, b]$. More precisely:

Claim. There is a subsequence $(x_{n_k})_k$ all of whose members are in $[a, b]$.

Proof of Claim. We define such a subsequence by induction as follows. For $k = 1$, taking $N := 1$, we get $n_1 \geq 1$ with $x_{n_1} \in [a, b]$. For $k = 2$, we take $N := n_1 + 1$ and get $n_2 \geq N > n_1$ with $x_{n_2} \in [a, b]$. Continue by induction: assuming n_k is defined, we take $N := n_k + 1$ and get $n_{k+1} \geq N > n_k$ with $x_{n_{k+1}} \in [a, b]$. Thus, we get a subsequence $(x_{n_k})_k$ all of whose members are in $[a, b]$. ⊣

The subsequence $(x_{n_k})_k$ is bounded because it is contained in $[a, b]$, so by the Bolzano–Weierstrass theorem, it has a convergent subsequence $(x_{n_{k_l}})_l$. Because each member of this subsequence is $\geq a$ and $\leq b$, so must be its limit because limits respect $\geq$ and $\leq$, as proven in class. Hence, $\lim_{l \to \infty} x_{n_{k_l}} \in [a, b]$. □

2. Suppose that for each $n \in \mathbb{N}$, $|x_{n+1} - x_n| \leq \frac{1}{2^n}$, and prove that $(x_n)_n$ is Cauchy. Deduce that $(x_n)_n$ converges.

Solution. For any natural numbers $m > n$, we can bound the distance $|x_n - x_m|$ using the distances between the consecutive members and the triangle inequality:

$$\begin{aligned}
|x_n - x_m| &\leq |x_n - x_{n+1}| + |x_{n+1} - x_{n+2}| + \ldots + |x_{m-1} - x_m| \\
&\leq \frac{1}{2^n} + \frac{1}{2^{n+1}} + \ldots + \frac{1}{2^{m-1}} = \frac{1}{2^n} \sum_{i=0}^{m-1-n} \frac{1}{2^i} \\
&< \frac{1}{2^n} \sum_{i=0}^{\infty} \frac{1}{2^i} = \frac{1}{2^n} \cdot \frac{1}{1 - \frac{1}{2}} = \frac{2}{2^n}.
\end{aligned}$$

But we know that $\frac{2}{2^n} \to 0$ (by the way, how do you prove this without using log?), so given an arbitrary $\varepsilon > 0$, there is an event $N \in \mathbb{N}$ such that $\forall n \geq N$, $\frac{2}{2^n} < \varepsilon$. Therefore, for all $m > n \geq N$,

$$|x_n - x_m| < \frac{2}{2^n} < \varepsilon.$$

This, by definition, means that $(x_n)_n$ is Cauchy, so by the Cauchy Convergence Criterion, it converges. □

3. Let $(x_k)_k$ be a sequence.

(a) Define another sequence $(a_n)_n$ such that for each $k \in \mathbb{N}$, x_k is the k^{th} partial sum of the series $\sum_{n=1}^{\infty} a_n$.

Solution. We need to find $(a_n)_n$ satisfying the following system of linear equations:

$$\begin{cases} a_1 = x_1 \\ a_1 + a_2 = x_2 \\ a_1 + a_2 + a_3 = x_3 \\ \vdots \\ a_1 + a_2 + \ldots + a_k = x_k \\ \vdots \end{cases}$$

Thus, put $a_1 := x_1$, $a_2 := x_2 - x_1$, $a_3 := x_3 - x_2$, and so on. More formally, by induction, supposing that x_{k-1} is indeed the $(k-1)^{\text{th}}$ partial sum, we see that $x_k = a_1 + a_2 + \ldots + a_k = (a_1 + a_2 + \ldots + a_{k-1}) + a_k = x_{k-1} + a_k$, which gives

$$a_k := x_k - x_{k-1}.$$

One can now verify, once again, that defining a_k this way indeed satisfies the desired property:

$$\begin{aligned} \sum_{n=1}^{k} a_n &= a_1 + a_2 + \ldots + a_k \\ &= x_1 + (x_2 - x_1) + (x_3 - x_2) + \ldots (x_{k-1} - x_{k-2}) + (x_k - x_{k-1}) = x_k. \end{aligned}$$

□

(b) Once again, suppose that for each $n \in \mathbb{N}$, $|x_{n+1} - x_n| \leq \frac{1}{2^n}$, and, using part (a) and the comparison test **for series**, prove that $(x_n)_n$ converges.

Hint: By part (a), the sequence $(x_k)_k$ converges if and only if $\sum_{n=1}^{\infty} a_n$ converges.

Solution. As the hint states, it is enough to prove that the series $\sum_{n=1}^{\infty} a_n$ converges. But note that $|a_{n+1}| = |x_{n+1} - x_n| \leq \frac{1}{2^n}$ and the series $\sum_{n=0}^{\infty} \frac{1}{2^n}$ converges, so by the Comparison Test for series, $\sum_{n=1}^{\infty} a_n$ also converges. □

4. Fix a real $\lambda > 1$.

(a) Prove that for any fixed integer $k \geq 0$, $\lim_{n\to\infty} \frac{n^k}{\lambda^n} = 0$.

Solution. We use the ratio test for sequences:

$$\frac{\frac{(n+1)^k}{\lambda^{n+1}}}{\frac{n^k}{\lambda^n}} = \frac{\lambda^n}{\lambda^{n+1}} \left(\frac{n+1}{n}\right)^k = \frac{1}{\lambda}\left(\frac{n+1}{n}\right)^k \to \frac{1}{\lambda} \cdot 1^k = \frac{1}{\lambda} < 1,$$

so, $\lim_{n\to\infty} \frac{n^k}{\lambda^n} = 0$. □

(b) Conclude that for any polynomial $p(x)$, $\lim_{n\to\infty} \frac{p(n)}{\lambda^n} = 0$.

Solution. Let $p(x) = a_d x^d + a_{d-1}x^{d-1} + \ldots + a_1 x + a_0$. Because the limit respects multiplication and addition, by part (a), for each $0 \le k \le d$, the sequence $\left(\frac{p(n)}{\lambda^n}\right)_n$ converges, we get

$$\lim_{n\to\infty} \frac{p(n)}{\lambda^n} = \lim_{n\to\infty} \sum_{k=0}^{d} \left(a_k \frac{n^k}{\lambda^n}\right) = \sum_{k=0}^{d} \left(a_k \lim_{n\to\infty} \frac{n^k}{\lambda^n}\right) = \sum_{k=0}^{d} (a_k \cdot 0) = 0.$$

□

(c) Also prove that for any polynomial $p(x)$, $\sum_{n=1}^{\infty} \frac{p(n)}{\lambda^n}$ converges.

Solution. Firstly, note that we cannot conclude the convergence of $\sum_{n=1}^{\infty} \frac{p(n)}{\lambda^n}$ from the fact that $\frac{p(n)}{\lambda^n} \to 0$ because the latter is only a necessary condition for convergence of series, but it is not sufficient (remember $\sum_{n=1}^{\infty} \frac{1}{n}$). Thus, we have to come up with something else.

Note that if $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ both converge, then so does $\sum_{n=1}^{\infty} (a_n + b_n)$ (why?); also, if $\sum_{n=1}^{\infty} a_n$ converges and $c \in \mathbb{R}$, then $\sum_{n=1}^{\infty} (ca_n)$ converges. Therefore, writing $p(x) = a_d x^d + a_{d-1}x^{d-1} + \ldots + a_1 x + a_0$, we see that it is enough to show that $\sum_{n=1}^{\infty} \frac{n^k}{\lambda^n}$ converges, for any fixed integer $k \ge 0$.

To this end, we use the ratio test for series. As calculated in part (a), $\frac{\frac{(n+1)^k}{\lambda^{n+1}}}{\frac{n^k}{\lambda^n}} \to \frac{1}{\lambda} < 1$, so the series converges. □

5. Let $(a_n)_n$ be a sequence of non-negative reals. Prove that if $\sum_{n=1}^{\infty} a_n$ converges, then for any $k \ge 1$, $\sum_{n=1}^{\infty} a_n^k$ also converges.

Solution. Which is bigger, a_n or a_n^k? This, of course, depends on whether $a_n \le 1$ or not, but we are not given this information. However, we know that $\sum_{n=1}^{\infty} a_n$ converges, which implies that $a_n \to 0$. Therefore, even though the first hundred million terms may be greater than 1, **eventually**, $a_n < 1$. Because the convergence of series doesn't depend on the first finitely many terms, we may assume without loss of generality that $a_n < 1$ for every $n \in \mathbb{N}$. Thus, $a_n^k \le a_n$ and the comparison test applies. □

6. (a) Suppose that $\sum_{n=1}^{\infty} a_n^2$ and $\sum_{n=1}^{\infty} b_n^2$ both converge, and prove that $\sum_{n=1}^{\infty} a_n b_n$ converges.

Solution.

Lemma. *For any $a, b \in \mathbb{R}$, $ab \leq \frac{a^2+b^2}{2}$.*

Proof. Follows from $a^2 + b^2 + 2ab = (a+b)^2 \geq 0$. ⊣

Because both $\sum_{n=1}^{\infty} a_n^2$ and $\sum_{n=1}^{\infty} b_n^2$ converge, so does the series $\sum_{n=1}^{\infty} \frac{1}{2}(a_n^2 + b_n^2)$. Therefore, by the lemma and the comparison test, the series $\sum_{n=1}^{\infty} a_n b_n$ also converges. □

(b) For each real $\lambda > 2$, construct an example of $(a_n)_n, (b_n)_n$ such that $\sum_{n=1}^{\infty} a_n^\lambda$ and $\sum_{n=1}^{\infty} b_n^\lambda$ both converge, but $\sum_{n=1}^{\infty} a_n b_n$ diverges.

Solution. Put $a_n = b_n := \frac{1}{n^{1/2}}$. Then, $a_n^\lambda = b_n^\lambda = \frac{1}{n^{\lambda/2}}$. Because $\lambda/2 > 1$, the series $\sum_{n=1}^{\infty} \frac{1}{n^{\lambda/2}}$ converges (we proved this in class). However, $a_n b_n = \frac{1}{n}$ and the series $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges. □

7. (Tricky) Let $(x_n)_n$ be sequence and $L \in \mathbb{R}$. Suppose that any subsequence $(x_{n_k})_k$ has a further subsequence $(x_{n_{k_l}})_l$ that converges to L. Prove that $(x_n)_n$ converges to L.

HINT: Prove the contrapositive. Assume that $(x_n)_n$ doesn't converge to L and build a subsequence "far" from L.

Solution. Suppose for contradiction that it is not true that $(x_n)_n$ converges to L. By definition, this means that there is $\varepsilon > 0$ such that for every $N \in \mathbb{N}$ (i.e. no matter how far we go) there is an index $n \geq N$ with $|x_n - L| \geq \varepsilon$. Using this, just like we did in the proof of Claim in Problem 1 above, we get a subsequence $(x_{n_k})_k$ such that $|x_{n_k} - L| \geq \varepsilon$ for every $k \in \mathbb{N}$. However, by our hypothesis, this subsequence must contain a further subsequence that converges to L, which is impossible (every term of that subsequence is at least ε distance away from L). □